

PARTIAL SEPARATRICES AND LOCAL BRUNELLA'S ALTERNATIVE

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ABSTRACT. Here we state a conjecture concerning a local version of Brunella's alternative: any codimension one foliation in $(\mathbb{C}^3, 0)$ without germ of invariant surface has a neighborhood of the origin formed by leaves containing a germ of analytic curve at the origin. We prove the conjecture for the class of codimension one foliations whose reduction of singularities is obtained by blowing-up points and curves of equireduction and such that the final singularities are free of saddle-nodes. The concept of "partial separatrix" for a given reduction of singularities has a central role in our argumentations, as well as the quantitative control of the generic Camacho-Sad index in dimension three. The "nodal components" are the only possible obstructions to get such germs of analytic curves. We use the partial separatrices to push the leaves near a nodal component towards compact diacritical divisors, finding in this way the desired analytic curves.

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1. INTRODUCTION

In this paper we improve the main statements in [11] concerning a local version of Brunella's alternative for germs of codimension one holomorphic foliations.

We know that any *non dicritical* germ of codimension one foliation \mathcal{F} in $(\mathbb{C}^3, 0)$ always has an invariant germ of analytic surface, as proved in [7] (the result is also true in higher ambient dimension [10]). Following a local version of Brunella's alternative [15] and a conjecture of D. Cerveau [14] we ask whether any germ of codimension one foliation \mathcal{F} over $(\mathbb{C}^3, 0)$ without invariant germ of surface satisfies the following property:

(\star) *There is an open neighborhood U of $0 \in \mathbb{C}^3$ such that any leaf of $\mathcal{F}|_U$ contains a germ of analytic curve at the origin.*

In view of the main result in [6], any germ of codimension one foliation \mathcal{F} in $(\mathbb{C}^3, 0)$ admits a reduction of singularities

$$\pi : (M, \pi^{-1}(0)) \rightarrow (\mathbb{C}^3, 0).$$

Using the arguments in [7], we see that if \mathcal{F} is without germ of invariant surface then there is a *compact dicritical component* D in the exceptional divisor E of π (this means that D is an irreducible surface contained in $\pi^{-1}(0)$ and transversal to the transformed foliation $\pi^*\mathcal{F}$).

We develop our study inside the class of germs of codimension one foliations of *Complex Hyperbolic* type, for short CH-foliations. We recall [11] that a germ of codimension one foliation \mathcal{F} in $(\mathbb{C}^n, 0)$ is a CH-foliation if for any generically transversal map

$$\phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^n, 0)$$

the transformed foliation $\phi^*\mathcal{F}$ has no saddle-nodes in its reduction of singularities, that is $\phi^*\mathcal{F}$ is a *generalized curve* in the sense of [2]. If $n = 3$, given a reduction of singularities π of \mathcal{F} , we have a CH-foliation if and only if there are no saddle-nodes among the singularities of $\pi^*\mathcal{F}$ of dimensional type two. We borrow the terminology of D. Cerveau in [13], where “complex hyperbolic” stands for simple singularities in dimension two that are not saddle-nodes.

Although in this paper we only consider a particular class of codimension one foliations, we believe that there are enough reasons to state the following conjecture:

“Any germ \mathcal{F} of CH-foliation on $(\mathbb{C}^3, 0)$ without germ of invariant analytic surface satisfies (\star) ”.

Our general strategy to prove the conjecture is to show that all the leaves “go” to a compact dicritical component after reduction of singularities. In fact, if L is a leaf of $\pi^*\mathcal{F}$ intersecting a compact dicritical component D at a point p , we can find a germ of analytic curve $(\tilde{\gamma}, p) \subset L$ and the image $(\pi(\tilde{\gamma}), 0)$ is the desired germ of analytic curve. As we have shown in [11], the main obstruction to following this strategy is the existence of a certain type of *uninterrupted nodal components*. They are a three-dimensional version of the “nodal separators” introduced by Mattei and Marín in [19]; they have also been recently considered by Camacho and Rosas [3] in the study of local minimal invariant sets in dimension two. Now, the natural procedure is to prove that any uninterrupted nodal component goes to a compact dicritical component, carrying the leaves with it, and thus it does not produce an obstruction to property (\star) . Indeed, it is necessary to assume that the foliation has no invariant germ of surface. We interpret this fact after reduction of singularities by observing that all the *partial separatrices* also go to a compact dicritical component.

The relationship between uninterrupted nodal components and partial separatrices is the main argument we use in this paper to obtain a proof of the conjecture for a particular class of CH-foliations on $(\mathbb{C}^3, 0)$.

Let us explain what are the *uninterrupted nodal components* and the *partial separatrices* for a given reduction of singularities π of a CH-foliation \mathcal{F} of $(\mathbb{C}^3, 0)$. First of all, we quickly recall the final situation after reduction of singularities [6, 7].

The exceptional divisor E of π is a normal crossings divisor and the singular locus $\text{Sing}\pi^*\mathcal{F}$ is a finite union of irreducible nonsingular curves having normal crossings with E . Any point $p \in \text{Sing}\pi^*\mathcal{F}$ has *dimensional type* $\tau_p \in \{2, 3\}$, which corresponds to the number of variables needed to locally describe the foliation.

If $\tau_p = 2$, there are local coordinates (x, y, z) at p such that $\pi^*\mathcal{F}$ is given by

$$(1) \quad \frac{dy}{y} - (\lambda + \phi(x, y)) \frac{dx}{x} = 0, \quad \phi(0, 0) = 0, \lambda \in \mathbb{C} \setminus \mathbb{Q}_{\geq 0}$$

and moreover $(x = 0) \subset E_{\text{inv}} \subset (xy = 0)$, where E_{inv} is the union of the invariant irreducible components of E . Note that $xy = 0$ are invariant surfaces for \mathcal{F} and that the singular locus $\text{Sing}\pi^*\mathcal{F}$ is $(x = y = 0)$ locally at p . The *transversal type* of $\pi^*\mathcal{F}$ at p is the germ of foliation \mathcal{T}_p in $(\mathbb{C}^2, 0)$ given by Equation (1).

Let Γ be the only irreducible curve of $\text{Sing}\pi^*\mathcal{F}$ passing through p . We know that $\mathcal{T}_p = \mathcal{T}_q$ for any $q \in \Gamma$ with $\tau_q = 2$. Thus $\mathcal{T}_p = \mathcal{T}_\Gamma$ is the *transversal type* of Γ . We say that Γ is *nodal* if $\lambda \in \mathbb{R}_{>0}$; in this case the transversal type is linearizable of the form $d(y/x^\lambda) = 0$. If $\lambda \in \mathbb{R}_{<0}$, we say that Γ is a *real saddle* and if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we say that Γ is a *complex saddle*.

At a point q of dimensional type three, the foliation $\pi^*\mathcal{F}$ is locally given by

$$\frac{dx}{x} + (\lambda + \phi(x, y, z)) \frac{dy}{y} + (\mu + \psi(x, y, z)) \frac{dz}{z} = 0$$

where $\phi(0, 0, 0) = \psi(0, 0, 0) = 0$ and $\lambda, \mu \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$, $\mu/\lambda \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$. Moreover

$$(xy = 0) \subset E_{\text{inv}} \subset (xyz = 0).$$

Note that the coordinate planes $xyz = 0$ are invariant surfaces and

$$\text{Sing}\pi^*\mathcal{F} = (x = y = 0) \cup (x = z = 0) \cup (y = z = 0).$$

Thus there are exactly three curves $\Gamma_1, \Gamma_2, \Gamma_3$ of $\text{Sing}\pi^*\mathcal{F}$ arriving at q . Up to reordering, we have the following five possibilities:

- (1) Γ_1, Γ_2 are nodal curves and Γ_3 is a real saddle.
- (2) Γ_1 is a nodal curve and Γ_2, Γ_3 are complex saddles.
- (3) Γ_1, Γ_2 and Γ_3 are real saddles.
- (4) Γ_1 is a real saddle and Γ_2, Γ_3 are complex saddles.
- (5) Γ_1, Γ_2 and Γ_3 are complex saddles.

We define an *uninterrupted nodal component* $\mathcal{N} \subset \text{Sing}\pi^*\mathcal{F}$ as any connected union of nodal curves such that at each point q of dimensional type three there are exactly two curves $\Gamma_1, \Gamma_2 \subset \mathcal{N}$ through q (we have the first case in the list above). We say that \mathcal{N} is *incomplete* if it intersects the compact dicritical part of the exceptional divisor. As we have seen in [11], if \mathcal{N} is incomplete the leaves “supported” by \mathcal{N} contain a germ of analytic curve. We have also obtained the following result:

Proposition 1 ([11]). *Consider a CH-foliation \mathcal{F} on $(\mathbb{C}^3, 0)$ without germ of analytic surface and let π be a reduction of singularities of \mathcal{F} . If any uninterrupted nodal component \mathcal{N} is incomplete, then \mathcal{F} satisfies (\star) .*

Thus, the conjecture is proved once we assure that there is a reduction of singularities such that any uninterrupted nodal component is incomplete.

Let us now introduce the concept of *partial separatrix*. We say that a curve $\Gamma \subset \text{Sing}\pi^*\mathcal{F}$ is a *trace curve* if it is contained in only one invariant irreducible component of the exceptional divisor E . Otherwise, the curve is the intersection of two invariant irreducible components of E and it is a *corner curve*. By definition, a *partial separatrix* C is any connected component of the union of trace curves. We say that C is *complete* if it does not intersect the compact dicritical part of E , otherwise, we say it is *incomplete*.

Following Cano-Cerveau's argumentations as in [7], given a partial separatrix C we find a germ of invariant surface

$$(S, C \cap \pi^{-1}(0)) \subset (M, \pi^{-1}(0))$$

supported by C . The inclusion above is closed if and only if C is complete. In this case we find by direct image a germ of surface $(\pi(S), 0)$ invariant for \mathcal{F} . Hence, we conclude:

If \mathcal{F} has no invariant germ of analytic surface, all the partial separatrices are incomplete.

The incomplete partial separatrices are the “guides” we use to take the uninterrupted nodal components to a compact dicritical component of the exceptional divisor. To do this, we need an accurate control of the transitions of the Camacho-Sad indices along the curves in the singular locus from one component of the exceptional divisor to another. This quantitative analysis focused on the partial separatrices is in contrast with the qualitative and combinatorial arguments we used in [11] to obtain the first results concerning the conjecture.

In this paper we prove the conjecture for the case of *special relatively isolated complex hyperbolic* germs \mathcal{F} of codimension one foliations in $(\mathbb{C}^3, 0)$. We precise the definitions in the next sections, but roughly speaking, this means that we can perform a reduction of singularities by blowing-up points until we reach a situation of equireduction along non compact curves, which we resolve by blowing-up only curves. This class of foliations contains both the cases of equireduction and the foliations associated to absolutely isolated singularities of surfaces. There are previous works on absolutely isolated singularities of vector fields [1] or on foliations desingularized by punctual blow-ups [9]; also, the results of Sancho de Salas in [22] concern these conditions very closely.

The main result of this paper is:

Theorem 1. *Any special relatively isolated CH-foliation \mathcal{F} in $(\mathbb{C}^3, 0)$ without germ of invariant analytic surface satisfies property (\star) .*

Theorem 1 improves the results in [11]. What we know from [11] is that if we take a complete nodal component \mathcal{N} , then the projection of \mathcal{N} contains at least one of the germs of curve of $\text{Sing}\mathcal{F}$ in $(\mathbb{C}^3, 0)$. In this way we have a criterion for the non existence of complete uninterrupted nodal components by looking at generic points of the germs of curve in $\text{Sing}\mathcal{F}$.

We prove Theorem 1 by showing that all the uninterrupted nodal components are incomplete. The argument is based on a control of the evolution of *incomplete points*. They are points such that there is a “local” partial separatrix over them which is incomplete. At the final step of the reduction of singularities, all the points are complete. We find a contradiction with the existence of a complete uninterrupted nodal component \mathcal{N} as follows. At the “birth level” of \mathcal{N} in the sequence of reduction of singularities, we find an incomplete point in a particular situation concerning the partial separatrices through it. We prove that this situation is part of a class of scenarios which persists along the reduction of singularities. In each scenario, there is at least one incomplete point. Then “a fortiori” we find an incomplete point at the last step of the reduction of singularities and obtain the desired contradiction.

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2. SPECIAL RELATIVELY ISOLATED CH-FOLIATIONS

A *special relatively isolated sequence* $\mathcal{S} = \{\pi_k\}_{k=1}^N$ of blow-ups of $(\mathbb{C}^3, 0)$ is a sequence

$$\mathcal{S} : (\mathbb{C}^3, 0) = (M_0, F_0) \xleftarrow{\pi_1} (M_1, F_1) \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} (M_N, F_N), \quad F_{k+1} = \pi_{k+1}^{-1}(F_k),$$

given by blow-ups π_{k+1} with center at closed germs $(Y_k, Y_k \cap F_k) \subset (M_k, F_k)$, such that for any $0 \leq k \leq N-1$ we have

- (1) $Y_k \cap F_k$ is a single point $Y_k \cap F_k = \{p_k\}$.
- (2) Y_k is either $\{p_k\}$ or a germ of nonsingular closed curve $(Y_k, p_k) \subset (M_k, F_k)$ having normal crossings with the exceptional divisor $E^k \subset M_k$ of

$$\sigma_k : (M_k, F_k) \rightarrow (\mathbb{C}^3, 0),$$

where $\sigma_k = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_k$.

- (3) If (Y_k, p_k) is a germ of curve, we have an *equireduction sequence* over (Y_k, p_k) in the following sense. For any $k < \ell \leq N-1$ we have one of the following situations:

- (a) $p_k \notin \pi_{k,\ell}(Y_\ell)$, where $\pi_{k,\ell} = \pi_{k+1} \circ \pi_{k+2} \circ \cdots \circ \pi_\ell$.
- (b) The center (Y_ℓ, p_ℓ) is a germ of curve and $\pi_{k,\ell}$ induces an isomorphism

$$\bar{\pi}_{k,\ell} : (Y_\ell, p_\ell) \rightarrow (Y_k, p_k).$$

Now, we say that a CH-foliation \mathcal{F} over $(\mathbb{C}^3, 0)$ is a *special relatively isolated CH-foliation* if there is a sequence \mathcal{S} of blow-ups as above such that

- i) For any $0 \leq k \leq N-1$ the center $Y_k \subset M_k$ of the blow-up π_{k+1} is contained in the locus $\text{Sing}^*(\mathcal{F}_k, E^k)$ of non simple points for \mathcal{F}_k, E^k , where $\mathcal{F}_k = \sigma_k^* \mathcal{F}$ is the transform of \mathcal{F} by σ_k .
- ii) All the points in F_N are simple points for \mathcal{F}_N, E^N . (See [6])

Since \mathcal{F} is a CH-foliation, the simple points after reduction of singularities are without saddle-nodes. Conversely, the fact that there is a reduction of singularities without saddle-nodes in the last step is enough to assure that \mathcal{F} is a CH-foliation. We refer to [11] for more details on the definitions.

Now, let us introduce some useful notations and remarks. The exceptional divisor $E^k \subset M_k$ of σ_k is a union of components

$$E^k = E_1^k \cup E_2^k \cup \cdots \cup E_k^k$$

where E_ℓ^k is the stric transform of E_ℓ^{k-1} for $1 \leq \ell < k$ and $E_k^k = \pi_k^{-1}(Y_k)$ is the exceptional divisor of the last blow-up π_k . Recall that M_k is a germ over the fiber $F_k = \sigma_k^{-1}(0)$. For any $k \geq 1$ we have that $F_k \subset E^k$ and the exceptional divisor E^k is a closed germ

$$(E^k, F_k) \subset (M_k, F_k).$$

A given irreducible component E_i^k may be an *invariant component*, if it is invariant for \mathcal{F}_k , or a *dicritical component*, when it is generically transversal to the foliation. We denote E_{inv}^k the union of the invariant components and E_{dic}^k the union of the dicritical ones.

Recall the definition of $\pi_{k,\ell} = \pi_{k+1} \circ \pi_{k+2} \circ \dots \circ \pi_\ell$. For certain special cases, we adopt the following simplified notations:

$$\begin{aligned} \sigma_k = \pi_{0,k} & : (M_k, F_k) \rightarrow (\mathbb{C}^3, 0) \\ \rho_k = \pi_{k,N} & : (M, F) \rightarrow (M_k, F_k) \\ \pi = \pi_{0,N} & : (M, F) \rightarrow (\mathbb{C}^3, 0). \end{aligned}$$

We denote the final step of the reduction of singularities by

$$M = M_N, E = E^N, \pi^* \mathcal{F} = \mathcal{F}_N, F = F_N = \pi^{-1}(0).$$

From now on, we fix a special relatively isolated CH-foliation \mathcal{F} of $(\mathbb{C}^3, 0)$ without germ of invariant analytic surface and a special relatively isolated sequence of blow-ups \mathcal{S} performing a reduction of singularities of \mathcal{F} .

If the first blow-up π_1 is centered at the origin $0 \in \mathbb{C}^3$, we have that the fiber F_k is the union E_C^k of the compact components of E^k , for any $1 \leq k \leq N$. As we shall explain later, the case when the first blow-up is centered in a germ of curve is of no interest to us, since in this situation the foliation \mathcal{F} has invariant surfaces. Thus, we also suppose along the paper that π_1 is a blow-up centered at the origin and hence $F_k = E_C^k$.

3. PARTIAL SEPARATRICES

Here we do a revision, adapted to our case, of the partial separatrices introduced in [11] and we briefly recall the global picture of a reduction of singularities (see [6] for more details).

Definition 1. A partial separatrix C for \mathcal{F}, π is any connected component of the union T of the trace curves of $\text{Sing}(\pi^* \mathcal{F})$. We say that C is complete if it does not intersect the union $E_{c,dic}$ of the compact dicritical components of E . We say that C is incomplete if it does intersect $E_{c,dic}$.

A partial separatrix C must be considered as a connected component of the germ $(T, T \cap F)$. So it is also a germ $(C, C \cap F)$. For shortness, we write C to denote the partial separatrix if there is no risk of confusion. The *compact part* $C \cap F$ of a partial separatrix is also connected and it is the union of the compact curves in C or just a single point.

Example 1. Let us recall Darboux-Jouanolou's example [18]. It is the conic foliation in $(\mathbb{C}^3, 0)$ given by the 1-form

$$\omega = (z^{m+1} - x^m y) dx + (x^{m+1} - y^m z) dy + (y^{m+1} - z^m x) dz.$$

The reduction of singularities consists of an initial dicritical blow-up of the origin followed by $m^2 + m + 1$ blow-ups centered at each of the lines of the singular locus

$$z^{m+1} - x^m y = x^{m+1} - y^m z = y^{m+1} - z^m x = 0.$$

We find $2(m^2 + m + 1)$ partial separatrices, all of them incomplete. Each one is a single non compact curve $(C^{(i)}, p_i)$, $i = 1, 2, \dots, 2(m^2 + m + 1)$ and hence the compact part $C^{(i)} \cap F$ is just the point p_i .

Let C be a partial separatrix and take a point $p \in C \cap F$. Recalling that the final singularities are complex hyperbolic, depending on the dimensional type $\tau = \tau(\pi^* \mathcal{F}; p)$ we find two situations:

- (1) If $\tau = 2$, there are coordinates x, y, z at p such that $E_{inv} = (x = 0)$, $E_{dic} \subset (z = 0)$ and

$$C = (x = y = 0) = \text{Sing}(\pi^*\mathcal{F}).$$

Moreover $S = (y = 0)$ is the only invariant germ of surface for $\pi^*\mathcal{F}$ at p not contained in E .

- (2) If $\tau = 3$, there are coordinates x, y, z at p such that $E = E_{inv} = (xy = 0)$,

$$C = (x = z = 0) \cup (y = z = 0),$$

and $\text{Sing}(\pi^*\mathcal{F}) = C \cup (x = y = 0)$. Moreover $S = (z = 0)$ is the only invariant germ of surface for $\pi^*\mathcal{F}$ at p not contained in E .

Gluing these situations along C as in [7], we find a germ of surface $(S_C, C \cap F)$ invariant for $\pi^*\mathcal{F}$ and not contained in E . Moreover, the inclusion of germs

$$(S_C, C \cap F) \subset (M, F)$$

is a closed immersion if and only if $S_C \cap F = C \cap F$. On the other hand, we have

$$S_C \cap F = C \cap F \Leftrightarrow C \cap E_{c,dic} = \emptyset.$$

That is, we obtain a closed immersion exactly when C is a complete partial separatrix. In this case, by Grauert's Theorem of the direct image under a proper morphism, we obtain a germ of surface $(\pi(S_C), 0)$ invariant for \mathcal{F} . We conclude:

Proposition 2. *If \mathcal{F} has no invariant germ of surface, then all the partial separatrices are incomplete.*

To finish this section, we give a result that justifies our assumption on the first blow-up being centered at the origin.

Proposition 3. *If the first blow-up is centered at a germ of curve γ , then \mathcal{F} has a germ of invariant surface.*

Proof. Note that we have equireduction along γ and thus the fiber $F = \pi^{-1}(0)$ is a union of compact curves. We have the following possible cases:

- (1) F contains a non invariant curve.
- (2) There is a dicritical component in the exceptional divisor, but all the curves in F are invariant.
- (3) All the components of the exceptional divisor are invariant and there is a curve Γ in F contained in the singular locus of $\pi^*\mathcal{F}$.
- (4) All the components of the exceptional divisor are invariant and the curves in F are not contained in the singular locus of $\pi^*\mathcal{F}$.

If there is a non invariant curve $\Gamma \subset F$, then Γ is necessarily contained in a dicritical component of E . Taking a generic point $p \in \Gamma$, the foliation $\pi^*\mathcal{F}$ is non singular at p and transversal to Γ . Thus, we find a germ of invariant surface (\tilde{S}, p) that gives a closed immersion into (M, F) and hence it projects onto a germ of surface $(S, 0)$ invariant for \mathcal{F} .

If all the curves of F are invariant but there is a dicritical component E_i of E , we consider the curve $\Gamma = E_i \cap F$. At a generic point p of Γ , there is a germ of surface (\tilde{S}, p) not contained in E such that $(\Gamma, p) = (\tilde{S} \cap E_i, p)$. By an extension of the argument of Cano-Cerveau [7] also used in [21] we can prolong (\tilde{S}, p) over the fiber F to find a closed immersion of a germ of invariant surface (\tilde{S}, F) in (M, F) . Finally, we project it by π to obtain a germ of surface $(S, 0)$ invariant for \mathcal{F} . This argument is also valid for the case (3).

Suppose now that all the irreducible components of the exceptional divisor E are invariant and the curves in F are not contained in the singular locus of $\pi^*\mathcal{F}$. This gives a non dicritical equireduction along γ in the sense of [5, 10]. In those papers it is proved that the reduction of singularities is given by the one of $\mathcal{F}|_\Delta$, where Δ is a plane transverse to γ . Then, any Camacho-Sad separatrix Σ of $\mathcal{F}|_\Delta$ induces a germ of surface $(S, 0)$ invariant for \mathcal{F} . \square

4. PARTIAL SEPARATRICES AT INTERMEDIATE STEPS

Let us give some remarks and definitions concerning the behavior of partial separatrices at an intermediate step (M_k, F_k) of the sequence \mathcal{S} of reduction of singularities, with $0 \leq k \leq N$.

Notation 1. If C is a partial separatrix, we denote $C_k = \rho_k(C)$. Let us remark that $C_k \cap F_k$ is a connected nonempty compact set and $\rho_k(C \cap F) = C_k \cap F_k$.

Consider an irreducible compact curve $\Gamma \subset M_k$ in the singular locus $\text{Sing} \mathcal{F}_k$. We have that $\Gamma \subset F_k$. By the properties of the sequence \mathcal{S} , only finitely many points of Γ will be modified in the further blow-ups $\pi_{k+1}, \pi_{k+2}, \dots, \pi_N$. Thus, there is a well defined strict transform $\Gamma' \subset M$ of Γ under ρ_k with $\Gamma' \subset \text{Sing} \pi^* \mathcal{F}$. Moreover, at a generic point $p \in \Gamma$ we have a point $p' \in \Gamma'$ where ρ_k induces an isomorphism

$$(M, p') \rightarrow (M_k, p).$$

In particular the pair \mathcal{F}_k, E_k has a simple singularity at such points p .

We say that Γ is a *trace curve*, respectively a *corner curve*, if and only if Γ' is so. If Γ is a trace curve, there is exactly one partial separatrix C such that $\Gamma' \subset C$. We say that C is the *partial separatrix associated to Γ* and we denote it by C_Γ .

Let us note that $C = C_\Gamma$ if and only if $\Gamma \subset C_k$.

Definition 2. Consider a point $p \in F_k$ and a partial separatrix C where $p \in C_k$. We say that C is complete at p if for any dicritical component E_i of E such that $E_i \subset \rho_k^{-1}(p)$ we have $E_i \cap C = \emptyset$. Otherwise we say that C is incomplete at p .

Remark 1. If C is complete at p , we find a closed immersion

$$(S_C, C \cap \rho_k^{-1}(p)) \subset (M, \rho_k^{-1}(p))$$

where $(S_C, C \cap \rho_k^{-1}(p))$ is a finite union of germs of surface invariant for $\pi^* \mathcal{F}$. Taking the image by ρ_k , we obtain a finite union

$$(\rho_k(S_C), p) \subset (M_k, p)$$

of germs of surface at p invariant for \mathcal{F}_k .

Remark 2. A partial separatrix C is complete, as stated in the introduction, if and only if it is complete at the origin $0 \in \mathbb{C}^3$. On the other hand, any partial separatrix C is complete at the points $p \in C \cap F$ in the final step of the reduction of singularities, even if p belongs to a compact dicritical component E_i of E .

Remark 3. Let $p_k \in F_k$ be a point such that the center Y_k of π_{k+1} is a germ of curve with $p_k \in Y_k$. In view of the equireduction properties of the sequence of reduction of singularities, we have that $\rho_k^{-1}(p_k)$ is a union of compact curves and hence it does not contain any component of E . Then any partial separatrix is complete at p_k .

Remark 4. We have $C_k = \pi_{k+1}(C_{k+1})$. If the partial separatrix C is complete at a point $p \in C_k \cap F_k$ then it is complete at all the points in $C_{k+1} \cap \pi_{k+1}^{-1}(p)$. Moreover, assume that π_{k+1} satisfies one of the following conditions:

- (1) The center Y_k of π_{k+1} does not contain p .
- (2) The center Y_k is a germ of curve.
- (3) The blow-up π_k is non dicritical.

Then the partial separatrix C is complete at $p \in C_k \cap F_k$ if and only if it is complete at all the points $p' \in C_{k+1} \cap \pi_{k+1}^{-1}(p)$.

Proposition 4. Let C be a partial separatrix complete at $p \in C_k \cap F_k$. We have:

- a) If π_{k+1} is centered at p , then π_{k+1} is a non-dicritical blow-up.
- b) If $p \in E_i^k$, where E_i^k is compact invariant, there is a compact trace curve $\Gamma \subset C_k \cap E_i^k$, with $p \in \Gamma$.
- c) If $p \in E_j^k$, where E_j^k is compact dicritical, then $C \cap E_j^N \neq \emptyset$.

Proof. We will do induction on $N - k$ to prove statements a), b) and c) in this order. If $k = N$, we are done. Assume that $k < N$. Since it is a local problem at p , if $p \notin Y_k$ we conclude by induction. Thus, we assume that $p \in Y_k$.

First case: the center of π_{k+1} is a germ of curve (Y_k, p) . We only have to prove b) and c). Note that for any compact component E_s^k such that $p \in E_s^k$ we have

$$\pi_{k+1}^{-1}(p) \subset E_s^{k+1}$$

and there is a point $p' \in C_{k+1} \cap \pi_{k+1}^{-1}(p)$. Assume b), we know that C is complete at p' and by induction hypothesis on $p' \in E_i^{k+1}$ we conclude that there is a trace compact curve $\Gamma' \subset E_i^{k+1} \cap C_{k+1}$ with $p' \in \Gamma'$. Now, it is enough to consider $\Gamma = \pi_{k+1}(\Gamma')$. Assume c), we know that C is complete at p' and by induction hypothesis on $p' \in E_j^{k+1}$ we conclude that $C \cap E_j^N \neq \emptyset$.

Second case: the center of π_{k+1} is the point p . We first prove a). If π_{k+1} is a dicritical blow-up, there is a point $p' \in C_{k+1} \cap E_{k+1}^{k+1}$. We know that C is complete at p' and by induction hypothesis we apply c) at p' to obtain that $C \cap E_{k+1}^N \neq \emptyset$. This contradicts the fact that C is complete at p .

Now we prove b) and c) already assuming that π_{k+1} is non-dicritical. Take a point $p' \in C_{k+1} \cap E_{k+1}^{k+1}$, we know that C is complete at p' . Since E_{k+1}^{k+1} is compact and invariant, by induction hypothesis on $p' \in E_{k+1}^{k+1}$, we find a trace compact curve $\Gamma'' \subset C_{k+1} \cap E_{k+1}^{k+1}$. Note that for any compact component E_s^k such that $p \in E_s^k$ we have that

$$E_s^{k+1} \cap E_{k+1}^{k+1}$$

is a projective line in the projective plane E_{k+1}^{k+1} . In particular there is at least one point $p_s'' \in \Gamma'' \cap E_s^{k+1}$. We know that C is complete at the point p_s'' . Assume b), we apply induction hypothesis on $p_s'' \in E_i^{k+1}$ to find a trace compact curve $\Gamma' \subset E_i^{k+1} \cap C_{k+1}$ such that $p_s'' \in \Gamma'$. We conclude by taking $\Gamma = \pi_{k+1}(\Gamma')$. Assume c), we apply induction hypothesis on $p_j'' \in E_j^{k+1}$ to find that $C \cap E_j^N \neq \emptyset$. \square

Remark 5. Proposition 4 can also be proved by invoking the germ of surface (S_C, p) obtained in Remark 1 and considering the intersections with the corresponding compact component of E^k . We have used the inductive arguments because of the general style of the paper.

Proposition 5. *Let C be an incomplete partial separatrix and consider an index $0 \leq k \leq N$. Then there is a point $p \in C_k$ such that C is not complete at p or there is a compact dicritical component E_j^k such that $C_k \cap E_j^k \neq \emptyset$.*

Proof. Induction on $N - k$. If $k = N$ we are done, since C intersects at least one compact dicritical component of the exceptional divisor. Take $k < N$. In order to find a contradiction, assume that C is complete at any $p \in C_k$ and that it does not intersect any compact dicritical component in E^k . We already know that C is complete at any point in C_{k+1} by Remark 4. By Proposition 4, we have that if E_{k+1}^{k+1} is a compact component with $E_{k+1}^{k+1} \cap C_{k+1} \neq \emptyset$, then E_{k+1}^{k+1} is an invariant component. This gives the desired contradiction by applying induction hypothesis. \square

Definition 3. *We say that $p \in F_k$ is an incomplete point if and only if there is a partial separatrix C such that $p \in C_k$ and C is incomplete at p .*

If there are no partial separatrices C such that $p \in C_k$ the point p is considered to be complete.

5. TRANSITION OF CAMACHO-SAD INDICES

Let us consider an irreducible compact curve $\Gamma \subset F_k \cap \text{Sing } \mathcal{F}_k$ and an invariant compact component E_i^k of E^k such that $\Gamma \subset E_i^k$. Note that, since Γ is compact, there are no dicritical components containing Γ .

Consider a plane section Δ transverse to Γ at a generic point $p \in \Gamma$. Taking appropriate local coordinates x, y at $p \in \Delta$, the restricted foliation $\mathcal{F}_k|_\Delta$ is given by a 1-form

$$\omega = x \left\{ (\lambda x + \mu y + \phi(x, y)) \frac{dx}{x} - dy \right\}$$

where $E_i^k \cap \Delta = (x = 0)$, $\mu \neq 0$ and $\phi(x, y)$ has a zero of order at least two at the origin. The Camacho-Sad index of $\mathcal{F}_k|_\Delta$ at p with respect to the invariant curve $x = 0$ is by definition the value $1/\mu$, see [4, 8]. We denote

$$\text{Ind}(\mathcal{F}, E_i; \Gamma) = \text{Ind}(\mathcal{F}_k|_\Delta, E_i^k \cap \Delta; p) = 1/\mu.$$

This index may be calculated in any step $k' \geq k$ of the reduction of singularities and at any point of the strict transform of Γ of dimensional type two.

Remark 6. Assume that Γ is contained in two compact invariant components E_i^k, E_j^k of E^k . We have that $\Gamma = E_i^k \cap E_j^k$. By the general properties of Camacho-Sad index [4], we have that

$$\text{Ind}(\mathcal{F}, E_i; \Gamma) \text{Ind}(\mathcal{F}, E_j; \Gamma) = 1.$$

Definition 4. Let $\Gamma \subset \text{Sing}(\pi^*\mathcal{F})$ be a compact curve contained in a compact invariant component E_i .

- (1) Γ is a nodal curve if and only if $\text{Ind}(\mathcal{F}, E_i; \Gamma) \in \mathbb{R}_{>0} \setminus \mathbb{Q}$.
- (2) Γ is a real saddle curve if and only if $\text{Ind}(\mathcal{F}, E_i; \Gamma) \in \mathbb{R}_{<0}$.
- (3) Γ is a complex saddle curve if and only if $\text{Ind}(\mathcal{F}, E_i; \Gamma) \in \mathbb{C} \setminus \mathbb{R}$.

Note that by Remark 6 the definition above does not depend on the invariant component E_i of E such that $\Gamma \subset E_i$.

Take a point $p \in E_i^k$ where E_i^k is a compact invariant component of E^k . We are interested in considering irreducible germs of curves

$$(\gamma, p) \subset (\text{Sing}\mathcal{F}_k \cap E_i^k, p)$$

Such a germ (γ, p) is contained in exactly one compact curve $\Gamma \subset \text{Sing}\mathcal{F}_k \cap E_i^k$. This allows us to put

$$\text{Ind}(\mathcal{F}, E_i; \gamma) = \text{Ind}(\mathcal{F}, E_i; \Gamma).$$

We denote $\mathcal{B}_i^k(p)$ the set of irreducible germs of curves $(\gamma, p) \subset (\text{Sing}\mathcal{F}_k \cap E_i^k, p)$.

In Proposition 6 we precise a relationship between the indices, counted with multiplicity, with respect to two incident compact components.

Proposition 6. Consider a point $p \in \Gamma = E_i^k \cap E_j^k$ where E_i^k and E_j^k are compact components of E^k . Assume that E_i^k is an invariant component of E^k .

- a) If E_j^k is a dicritical component and $\mathcal{G} = \mathcal{F}_k|_{E_j^k}$ we have

$$\text{Ind}(\mathcal{G}, \Gamma; p) = \sum_{\gamma \in \mathcal{B}_i^k(p)} (\gamma, \Gamma)_p \text{Ind}(\mathcal{F}, E_i; \gamma),$$

where $(\gamma, \Gamma)_p$ is the intersection multiplicity of γ, Γ at p .

- b) If E_j^k is an invariant component and $\alpha = \text{Ind}(\mathcal{F}, E_i; \Gamma)$, we have

$$\sum_{\gamma \in \mathcal{B}_i^k(p) \setminus \{\Gamma\}} (\gamma, \Gamma)_p \text{Ind}(\mathcal{F}, E_i; \gamma) = -\alpha \sum_{\delta \in \mathcal{B}_j^k(p) \setminus \{\Gamma\}} (\delta, \Gamma)_p \text{Ind}(\mathcal{F}, E_j; \delta).$$

Proof. We do induction on $N - k$. Let us consider first the case $k = N$:

- (1) If p is non singular, it belongs to at most one invariant component of the divisor. We have a) with $\mathcal{B}_i^k(p) = \emptyset$. Thus we are done.
- (2) Assume that p is of dimensional type two and E_j is a dicritical component. Then the singular locus is non singular at p and E_j gives a section transversal to it. We are done by the definition of the generic index.
- (3) Assume p is of dimensional type two and E_j is invariant. The singular locus is Γ . In this case $\mathcal{B}_i^k(p) = \mathcal{B}_j^k(p) = \{\Gamma\}$ and there is nothing to prove.

- (4) Assume that p is of dimensional type three. Then E_j is necessarily invariant and there are local coordinates x, y, z at p such that

$$E_i = (x = 0), \quad E_j = (y = 0), \quad \Gamma = (x = y = 0),$$

the plane $z = 0$ is invariant and the singular locus is given by $\Gamma \cup \gamma \cup \delta$, where

$$\gamma = (x = z = 0); \quad \delta = (y = z = 0).$$

Moreover, the foliation \mathcal{F} is given locally at p by an integrable 1-form of the type

$$\omega = \frac{dx}{x} + (-\alpha + b(x, y, z)) \frac{dy}{y} + (-\beta + c(x, y, z)) \frac{dz}{z}, \quad \alpha \neq 0 \neq \beta.$$

By the integrability condition $\omega \wedge d\omega = 0$, we have

$$\begin{aligned} b(x, y, z) &= xb'(x, y, z) + yb''(x, y, z) \\ c(x, y, z) &= xc'(x, y, z) + yc''(x, y, z) \end{aligned}$$

and thus

$$\frac{-\beta + c(x, y, z)}{-\alpha + b(x, y, z)} = \frac{\beta}{\alpha} + yf'(x, y, z) + zf''(x, y, z).$$

Then, we have

$$\text{Ind}(\mathcal{F}, E_i; \Gamma) = \alpha, \quad \text{Ind}(\mathcal{F}, E_i; \gamma) = \beta, \quad \text{Ind}(\mathcal{F}, E_j; \delta) = -\beta/\alpha.$$

The desired relation is $\beta = -\alpha(-\beta/\alpha)$, that is obviously satisfied.

Now, suppose that $k < N$. If $p \notin Y_k$ we are done by induction; hence we assume $p \in Y_k$. Moreover, the center Y_k of the blow-up π_{k+1} cannot be a germ of curve, since there are two compact components of E^k through p and Y_k should have normal crossings with E^k . Thus $Y_k = \{p\}$.

Let us give some remarks and fix notations. We put

$$\Gamma' = E_i^{k+1} \cap E_j^{k+1}, \quad L'_i = E_i^{k+1} \cap E_{k+1}^{k+1}, \quad L'_j = E_j^{k+1} \cap E_{k+1}^{k+1}, \quad p' = \Gamma' \cap L'_i.$$

In view of Noether's formula for the intersection multiplicity (see [17] for instance), given $\gamma \in \mathcal{B}(i; p) \setminus \{\Gamma\}$ we have

$$(\gamma, \Gamma)_p = (\gamma', \Gamma')_{p'} + \sum_{q \in L'_i} (\gamma', L'_i)_q$$

where γ' stands for the strict transform of γ . Let us also note that

$$\text{Ind}(\mathcal{F}, E_i; \gamma) = \text{Ind}(\mathcal{F}, E_i; \gamma'),$$

since the computations are made at generic points of $\gamma \subset E_i^k$.

On the other hand, note that E_{k+1}^{k+1} is a projective plane and $L'_i, L'_j \subset E_{k+1}^{k+1}$ are both projective lines. In particular, let $\Lambda \subset E_{k+1}^{k+1}$ be a global irreducible curve $\Lambda \subset \text{Sing}(\mathcal{F}_{k+1})$ of degree d_Λ with $\Lambda \neq L'_i, L'_j$. By Bezout's Theorem, we know that

$$d_\Lambda = \sum_{q \in L'_i; q \in \delta \subset \Lambda} (\delta, L'_i)_q = \sum_{q \in L'_j; q \in \delta \subset \Lambda} (\delta, L'_j)_q,$$

where δ runs over the irreducible branches of Λ at q .

Now, we have four cases to consider:

- i) E_j^k is dicritical and π_{k+1} is a dicritical blow-up.
- ii) E_j^k is dicritical and π_{k+1} is a non dicritical blow-up.
- iii) E_j^k is invariant and π_{k+1} is a dicritical blow-up.
- iv) E_j^k is invariant and π_{k+1} is a non dicritical blow-up.

Assume first that E_j^k is a dicritical component. Let us note that

$$\Gamma \not\subset \text{Sing}(\mathcal{F}_k); \Gamma', L'_j \not\subset \text{Sing}(\mathcal{F}_{k+1}).$$

The induced induced foliation \mathcal{G}' by \mathcal{F}_{k+1} on E_j^{k+1} is the transform of \mathcal{G} by the restriction

$$\tilde{\pi}_{k+1} : E_j^{k+1} \rightarrow E_j^k$$

of the blow-up π_{k+1} . In particular, by the known properties of Camacho-Sad index (see [4, 8]) we have that

$$\text{Ind}(\mathcal{G}, \Gamma; p) = \text{Ind}(\mathcal{G}', \Gamma'; p') + 1.$$

First case: π_{k+1} is a dicritical blow-up. Let us denote \mathcal{G}_1 the induced foliation by \mathcal{F}_{k+1} on E_{k+1}^{k+1} . The self-intersection of the projective line L'_i in the projective plane E_{k+1}^{k+1} is equal to +1. Then we have

$$\sum_{q \in L'_i} \text{Ind}(\mathcal{G}_1, L'_i; q) = +1,$$

Let us note that since Γ', L'_i are not in the singular locus we have a bijection

$$\mathcal{B}_i^k(p) \leftrightarrow \bigcup_{q \in L'_i} \mathcal{B}_i^{k+1}(q)$$

given by the strict transform $\gamma \mapsto \gamma'$. Applying induction hypothesis to the points of L'_i we deduce that

$$\sum_{q \in L'_i; \gamma \in \mathcal{B}_i^k(p)} (\gamma', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \gamma) = \sum_{q \in L'_i} \text{Ind}(\mathcal{G}_1, L'_i; q) = +1.$$

Applying induction hypothesis at p' as well, we have

$$\begin{aligned} & \sum_{\gamma \in \mathcal{B}_i^k(p)} (\gamma, \Gamma)_p \text{Ind}(\mathcal{F}, E_i; \gamma) = \\ &= \sum_{\gamma \in \mathcal{B}_i^k(p)} \left((\gamma', \Gamma')_{p'} + \sum_{q \in L'_i} (\gamma', L'_i)_q \right) \text{Ind}(\mathcal{F}, E_i; \gamma) = \\ &= \text{Ind}(\mathcal{G}', \Gamma'; p') + 1 = \text{Ind}(\mathcal{G}, \Gamma; p). \end{aligned}$$

This case is ended.

Second case: π_{k+1} is a non dicritical blow-up. Let us denote $\tilde{\alpha} = \text{Ind}(\mathcal{F}, E_i; L'_i)$. By induction hypothesis at p' we have

$$\text{Ind}(\mathcal{G}', \Gamma'; p') = \tilde{\alpha} + \sum_{\gamma \in \mathcal{B}_i^k(p)} (\gamma', \Gamma')_{p'} \text{Ind}(\mathcal{F}, E_i; \gamma'),$$

and since $\text{Ind}(\mathcal{G}', \Gamma'; p') = \text{Ind}(\mathcal{G}, \Gamma; p) - 1$, we can put

$$\text{Ind}(\mathcal{G}, \Gamma; p) = \tilde{\alpha} + 1 + \sum_{\gamma \in \mathcal{B}_i^k(p)} (\gamma', \Gamma')_{p'} \text{Ind}(\mathcal{F}, E_i; \gamma').$$

By Noether's Theorem we have $(\gamma', \Gamma')_{p'} = (\gamma, \Gamma)_p - \sum_{q \in L'_i} (\gamma', L'_i)_q$. Then

$$\text{Ind}(\mathcal{G}, \Gamma; p) = \sum_{\gamma \in \mathcal{B}_i^k(p)} (\gamma, \Gamma)_p \text{Ind}(\mathcal{F}, E_i; \gamma) + \beta$$

where

$$\beta = \tilde{\alpha} + 1 - \sum_{q \in L'_i} \sum_{\gamma \in \mathcal{B}_i^k(p)} (\gamma', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \gamma').$$

Now, it is enough to show that $\beta = 0$. By induction hypothesis in the statement b) referred to E_i^{k+1} and E_{k+1}^{k+1} we have that

$$\sum_{q \in L'_i; \gamma \in \mathcal{B}_i^k(p)} (\gamma', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \gamma') = -\tilde{\alpha} \sum_{\Lambda \subset E_{k+1}^{k+1}, \Lambda \neq L'_i} d_\Lambda \text{Ind}(\mathcal{F}, E_{k+1}; \Lambda).$$

Now, we apply induction hypothesis in the statement a) referred to E_{k+1}^{k+1} and E_j^{k+1} to obtain

$$\sum_{q \in L'_j} \text{Ind}(\mathcal{G}', L'_j; q) = 1/\tilde{\alpha} + \sum_{\Lambda \subset E_{k+1}^{k+1}; \Lambda \neq L'_i} d_\Lambda \text{Ind}(\mathcal{F}, E_{k+1}; \Lambda),$$

where $1/\tilde{\alpha} = \text{Ind}(\mathcal{F}, E_{k+1}; L'_i)$. Recalling that the self intersection of L'_j in E_j^{k+1} is equal to -1 , we have $-1 = \sum_{q \in L'_j} \text{Ind}(\mathcal{G}', L'_j; q)$ and we obtain

$$-1 = 1/\tilde{\alpha} - (1/\tilde{\alpha}) \sum_{q \in L'_i; \gamma \in \mathcal{B}_i^k(p)} (\gamma', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \gamma').$$

That is

$$\beta = \tilde{\alpha} + 1 - \sum_{q \in L'_i; \gamma \in \mathcal{B}_i^k(p)} (\gamma', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \gamma') = 0$$

and we are done.

Let us suppose finally that E_j^k is an invariant component.

First case: π_{k+1} is a dicritical blow-up. Let \mathcal{G}_1 be the induced foliation by \mathcal{F}_{k+1} on E_{k+1}^{k+1} . By applying induction hypothesis at L'_i and L'_j and recalling that the self-intersection of L'_i, L'_j inside E_{k+1}^{k+1} is $+1$, we have

$$1 = \sum_{q \in L'_i} \text{Ind}(\mathcal{G}_1, L'_i; q) = \sum_{q \in L'_j} \text{Ind}(\mathcal{G}_1, L'_j; q),$$

and hence

$$(2) \quad 1 = \alpha + \sum_{q \in L'_i} \sum_{\gamma \in \mathcal{B}_i^k(p) \setminus \{\Gamma\}} (\gamma', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \gamma'),$$

$$(3) \quad 1 = (1/\alpha) + \sum_{q \in L'_j} \sum_{\delta \in \mathcal{B}_j^k(p) \setminus \{\Gamma\}} (\delta', L'_j)_q \text{Ind}(\mathcal{F}, E_j; \delta').$$

Also, by induction hypothesis at p' referred to E_i^{k+1} and E_j^{k+1} we have

$$(4) \quad \sum_{\gamma \in \mathcal{B}_i^k(p) \setminus \{\Gamma\}} (\gamma', \Gamma')_{p'} \text{Ind}(\mathcal{F}, E_i; \gamma') = -\alpha \sum_{\delta \in \mathcal{B}_j^k(p) \setminus \{\Gamma\}} (\delta', \Gamma')_{p'} \text{Ind}(\mathcal{F}, E_j; \delta').$$

Using Noether's formula and the equalities (2, 3), we have

$$\begin{aligned} \sum_{\gamma \in \mathcal{B}_i^k(p) \setminus \{\Gamma\}} (\gamma, \Gamma)_p \text{Ind}(\mathcal{F}, E_i; \gamma) &= \sum_{\gamma \in \mathcal{B}_i^k(p) \setminus \{\Gamma\}} (\gamma', \Gamma')_{p'} \text{Ind}(\mathcal{F}, E_i; \gamma') + (1 - \alpha), \\ -\alpha \sum_{\delta \in \mathcal{B}_j^k(p) \setminus \{\Gamma\}} (\delta, \Gamma)_p \text{Ind}(\mathcal{F}, E_i; \delta) &= -\alpha \sum_{\gamma \in \mathcal{B}_j^k(p) \setminus \{\Gamma\}} (\delta', \Gamma')_{p'} \text{Ind}(\mathcal{F}, E_i; \delta') + (1 - \alpha) \end{aligned}$$

and we are done by Equation (4).

Second case: π_{k+1} is a non dicritical blow-up. Let us denote

$$\beta = \text{Ind}(\mathcal{F}, E_i; L'_i); \quad \rho = \text{Ind}(\mathcal{F}, E_j; L'_j).$$

We have $1/\beta = \text{Ind}(\mathcal{F}, E_{k+1}; L'_i)$ and $1/\rho = \text{Ind}(\mathcal{F}, E_{k+1}; L'_j)$. Let us put

$$\epsilon = \sum_{\Lambda \subset E_{k+1}^{k+1}, \Lambda \neq L'_i, L'_j} d_\Lambda \text{Ind}(\mathcal{F}, E_{k+1}; \Lambda).$$

Now, if we take a generic plane section Δ at p and we apply Camacho-Sad's equality to $\mathcal{F}_k|_\Delta$ after the blow-up π_{k+1} , we obtain

$$-1 = 1/\beta + 1/\rho + \epsilon.$$

By induction hypothesis referred to E_i^{k+1} and E_{k+1}^{k+1} , we have the following equality

$$\alpha + \sum_{q \in L'_i} \sum_{\gamma \in \mathcal{B}_i^k(p) \setminus \{\Gamma\}} (\gamma', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \gamma') = -\beta((1/\rho) + \epsilon) = \beta + 1$$

and thus

$$\sum_{q \in L'_i} \sum_{\gamma \in \mathcal{B}_i^k(p) \setminus \{\Gamma\}} (\gamma', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \gamma') = -\alpha + \beta + 1.$$

Now, applying induction referred to E_j^{k+1} and E_{k+1}^{k+1} , we have

$$(1/\alpha) + \sum_{q \in L'_i} \sum_{\delta \in \mathcal{B}_j^k(p) \setminus \{\Gamma\}} (\delta', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \delta') = -\rho((1/\beta) + \epsilon) = \rho + 1$$

and thus

$$-\alpha \sum_{q \in L'_i} \sum_{\delta \in \mathcal{B}_j^k(p) \setminus \{\Gamma\}} (\delta', L'_i)_q \text{Ind}(\mathcal{F}, E_i; \delta') = 1 - \alpha(\rho + 1).$$

Applying induction hypothesis at p' , we have

$$\begin{aligned} & \beta + \sum_{\gamma \in \mathcal{B}_i^k(p) \setminus \{\Gamma\}} (\gamma', \Gamma')_{p'} \text{Ind}(\mathcal{F}, E_j; \gamma') = \\ & = -\alpha \left(\rho + \sum_{\delta \in \mathcal{B}_j^k(p) \setminus \{\Gamma\}} (\delta', \Gamma')_{p'} \text{Ind}(\mathcal{F}, E_j; \delta') \right). \end{aligned}$$

Thus, by Noether's equality, we only have to verify that

$$(-\alpha + \beta + 1) - \beta = (1 - \alpha(\rho + 1)) + \alpha\rho$$

and this is evident. \square

Corollary 1. *Let $p \in F_k$ be a point such that $p \in E_i^k \cap E_j^k \cap E_\ell^k$ where E_i^k , E_j^k and E_ℓ^k are compact invariant components of E^k . Let us denote*

$$\begin{aligned} \Gamma_\ell &= E_i^k \cap E_j^k, \Gamma_j = E_i^k \cap E_\ell^k, \Gamma_i = E_j^k \cap E_\ell^k. \\ \alpha &= \text{Ind}(\mathcal{F}, E_i; \Gamma_\ell), \beta = \text{Ind}(\mathcal{F}, E_i; \Gamma_j), \rho = \text{Ind}(\mathcal{F}, E_j; \Gamma_i). \end{aligned}$$

Then, we have $\beta = -\alpha\rho$.

Proof. Induction on $N - k$. If $k = N$, we are done (see (4) in the proof of the case $k = N$ in Proposition 6). Assume that $k < N$ and $p \in Y_k$ as in previous proofs. We have that $Y_k = \{p\}$. If the blow-up is non dicritical, we put

$$\nu = \text{Ind}(\mathcal{F}, E_i; E_i \cap E_{k+1}), \xi = \text{Ind}(\mathcal{F}, E_\ell; E_\ell \cap E_{k+1}), \mu = \text{Ind}(\mathcal{F}, E_j; E_j \cap E_{k+1}).$$

By induction hypothesis, we have $\nu = -\alpha\mu$, $\nu = -\xi\beta$, $\mu = -\xi\rho$. That is, we have $\xi\beta = \alpha\mu = -\alpha\xi\rho$ and thus $\beta = -\alpha\rho$.

Assume now that the blow-up is dicritical. We denote by $\Gamma'_\ell, \Gamma'_j, \Gamma'_i$ the strict transforms of $\Gamma_\ell, \Gamma_j, \Gamma_i$ respectively. We also denote by

$$\mathcal{B}_i^* = \mathcal{B}_i^k(p) \setminus \{\Gamma_\ell, \Gamma_j\}; \quad \mathcal{B}_j^* = \mathcal{B}_j^k(p) \setminus \{\Gamma_\ell, \Gamma_i\}; \quad \mathcal{B}_\ell^* = \mathcal{B}_\ell^k(p) \setminus \{\Gamma_i, \Gamma_j\};$$

and we put

$$I_u^{(v)} = \sum_{\gamma \in \mathcal{B}_u^*} (\Gamma_v, \gamma)_p \text{Ind}(\mathcal{F}, E_u; \gamma)$$

for $u \neq v$ with $u, v \in \{i, j, \ell\}$. Given a germ of curve γ we denote by γ' the strict transform of γ , as usual. Take also the following notations

$$p'_u = \Gamma'_u \cap E_{k+1}^{k+1}, \quad L'_u = E_u^{k+1} \cap E_{k+1}^{k+1}; \quad u \in \{i, j, \ell\}$$

and

$$I'_u = \sum_{q' \in L'_u, \gamma \in \mathcal{B}_u^*} (L'_u, \gamma')_{q'} \text{Ind}(\mathcal{F}, E_i; \gamma); \quad u \in \{i, j, \ell\}.$$

Finally, we put

$$I_u^{(v)} = \sum_{\gamma \in \mathcal{B}_u^*} (\Gamma'_v, \gamma')_{p'_v} \text{Ind}(\mathcal{F}, E_u; \gamma')$$

for $u \neq v$ with $u, v \in \{i, j, \ell\}$.

By Noether's formula, we have

$$I_u^{(v)} = I'_u + I_i^{(v)}, \quad \text{for } u, v \in \{i, j, \ell\}, u \neq v.$$

Now, by applying part a) of Proposition 6 to the exceptional divisor and each of three other divisors, we have

$$(5) \quad \alpha + \beta + I'_i = \frac{1}{\beta} + \frac{1}{\rho} + I'_\ell = \frac{1}{\alpha} + \rho + I'_j = 1.$$

By Proposition 6 we have

$$\begin{aligned} I_i^{(\ell)} &= -\alpha I_j^{(\ell)}; \quad I_i^{(j)} = -\beta I_\ell^{(j)}; \quad I_\ell^{(i)} = -(1/\rho) I_j^{(i)}. \\ I_i^{(\ell)} &= -\alpha I_j^{(\ell)}; \quad I_i^{(j)} = -\beta I_\ell^{(j)}; \quad I_\ell^{(i)} = -(1/\rho) I_j^{(i)}. \end{aligned}$$

We deduce that

$$I'_i = -\alpha I'_j; \quad I'_i = -\beta I'_\ell; \quad I'_\ell = -(1/\rho) I'_j.$$

This implies that

$$-\alpha I'_j = \beta(1/\rho) I'_j.$$

Hence, if $I'_j \neq 0$ we conclude that $\beta = -\alpha\rho$ and we are done.

Assume now that $I'_j = 0$ and hence $I'_i = I'_j = I'_\ell = 0$. By Equation 5 we have

$$\alpha + \beta = (1/\beta) + (1/\rho) = (1/\alpha) + \rho = 1.$$

We deduce that $1 + \alpha\rho = \alpha$, hence $1 = \alpha(1 - \rho)$, but $1 = \alpha + \beta$. This implies that $\beta = -\alpha\rho$ as desired. \square

Let us consider a point p in a compact invariant component E_i^k of E^k and a partial separatrix C . We denote by $\mathcal{B}_i^k(C; p)$ the set of germs of curve (γ, p) such that $(\gamma, p) \subset (C_k \cap E_i^k, p)$.

Corollary 2. *Let us consider a point $p \in \Gamma = E_i^k \cap E_j^k$, where E_i^k and E_j^k are compact invariant components of E^k and a partial separatrix C . Assume that p is a complete point for C . We have*

$$\sum_{\gamma \in \mathcal{B}_i^k(C; p)} (\gamma, \Gamma)_p \text{Ind}(\mathcal{F}, E_i^k; \gamma) = -\alpha \sum_{\eta \in \mathcal{B}_j^k(C; p)} (\eta, \Gamma)_p \text{Ind}(\mathcal{F}, E_j^k; \eta).$$

where $\alpha = \text{Ind}(\mathcal{F}, E_i^k; \Gamma)$.

Proof. We do induction on $N - k$ as usual. If $k = N$ we are done, by the local expression at simple points. Assume that $k < N$. We suppose without loss of generality that $p \in Y_k$ and thus the next blow-up π_{k+1} is centered at the point p . Since C is complete at p , the blow-up is non-dicritical. Put

$$\Gamma' = E_i^{k+1} \cap E_j^{k+1}, \quad L'_i = E_{k+1}^{k+1} \cap E_i^{k+1}, \quad L'_j = E_{k+1}^{k+1} \cap E_j^{k+1}$$

and let $p' = \Gamma' \cap E_{k+1}^{k+1}$. Denote as usual by γ' the strict transform of the germ of curve γ and put

$$\begin{aligned} I'_u(C) &= \sum_{q \in L'_u} \sum_{\gamma \in \mathcal{B}_u^k(C; p)} (\gamma', L'_u)_q \text{Ind}(\mathcal{F}, E_u^k; \gamma), \\ I''_u(C) &= \sum_{\gamma \in \mathcal{B}_u^k(C; p)} (\gamma', \Gamma')_{p'} \text{Ind}(\mathcal{F}, E_u^k; \gamma) \end{aligned}$$

for $u \in \{i, j\}$. We have that

$$\sum_{\gamma \in \mathcal{B}_u^k(C; p)} (\gamma, \Gamma)_p \text{Ind}(\mathcal{F}, E_u^k; \gamma) = I'_u(C) + I''_u(C), \quad u \in \{i, j\}$$

and by induction hypothesis we know that $I''_i(C) = -\alpha I''_j(C)$. Let us denote

$$\beta = \text{Ind}(\mathcal{F}, E_i^{k+1}; L'_i); \quad \rho = \text{Ind}(\mathcal{F}, E_j^{k+1}; L'_j).$$

Also by induction hypothesis, we have

$$(-1/\beta)I'_i(C) = \sum_{\Lambda} d_{\Lambda} \text{Ind}(\mathcal{F}, E_{k+1}^{k+1}; \Lambda) = (-1/\rho)I'_j(C),$$

where Λ stands for the global irreducible curves $\Lambda \subset E_{k+1}^{k+1}$ such that $\Lambda \subset C_{k+1}$. Applying Corollary 1, we deduce that

$$I'_i(C) = (\beta/\rho)I'_j(C) = -\alpha I'_j(C)$$

and we are done. \square

6. INDICES OF PARTIAL SEPARATRICES

Consider a partial separatrix C . Here we show that it is possible to define the index $\text{Ind}(C; E_i)$ relative to any invariant compact component E_i of E . Given an invariant compact component E_i of E , we denote by $\mathcal{B}_i C$ the set of global irreducible curves in $C \cap E_i$. We put $\text{Ind}(C; E_i) = 0$ if there is no compact curve of C contained in E_i . Otherwise, we shall put

$$\text{Ind}(C; E_i) = \text{Ind}(\mathcal{F}, E_i; \Gamma)$$

where $\Gamma \in \mathcal{B}_i C$. Proposition 7 assures that the definition is consistent.

Proposition 7. *Let C be a partial separatrix and E_i a compact invariant component of E . If $\Gamma_1, \Gamma_2 \in \mathcal{B}_i C$, we have $\text{Ind}(\mathcal{F}, E_i; \Gamma_1) = \text{Ind}(\mathcal{F}, E_i; \Gamma_2)$.*

Before giving the proof of Proposition 7, let us introduce the *dual graph* \mathcal{G}_N of the compact invariant components. This graph has vertices corresponding to the compact invariant components; two vertices E_i, E_j are joined by a wedge if and only if $E_i \cap E_j \neq \emptyset$. It is the last one of the series of dual graphs \mathcal{G}_k of the compact invariant components of E^k .

Since each new invariant compact component is produced by the blow-up of a point, we see that given two compact invariant components E_i and E_j we have that either $E_i \cap E_j = \emptyset$ or $E_i \cap E_j$ is an irreducible compact curve.

The graph \mathcal{G}_{k+1} is obtained from \mathcal{G}_k as follows. If the blow-up π_{k+1} is dicritical, then $\mathcal{G}_{k+1} = \mathcal{G}_k$. If the center of π_{k+1} is a curve, we also have that $\mathcal{G}_{k+1} = \mathcal{G}_k$. If π_{k+1} is non dicritical and the center is a point p_k , we have four possibilities:

- (1) The point p_k does not belong to any invariant compact component of E^k . In this case, the graph \mathcal{G}_{k+1} is obtained from \mathcal{G}_k by adding a new connected component to \mathcal{G}_k consisting in a single vertex that represents the exceptional divisor E_{k+1}^{k+1} . No new wedges are added.
- (2) The point p_k belongs to a single invariant compact component E_i^k of E^k . Then \mathcal{G}_{k+1} is obtained from \mathcal{G}_k by adding a new vertex that represents the exceptional divisor E_{k+1}^{k+1} and a new wedge connecting it with E_i^{k+1} .
- (3) The point p_k belongs to exactly two invariant compact components E_i^k, E_j^k of E^k . Then \mathcal{G}_{k+1} is obtained from \mathcal{G}_k by adding a new vertex that represents the exceptional divisor E_{k+1}^{k+1} and two new wedges connecting it with E_i^{k+1} and E_j^{k+1} .
- (4) The point p_k belongs to three invariant compact components E_i^k, E_j^k, E_{ℓ}^k of E^k . Then \mathcal{G}_{k+1} is obtained from \mathcal{G}_k by adding a new vertex that represents the exceptional divisor E_{k+1}^{k+1} and three new wedges connecting it with E_i^{k+1} , E_j^{k+1} and E_{ℓ}^{k+1} .

A *chain* of length s in \mathcal{G}_N is any sequence

$$(6) \quad c = (E_{i_0}, w_1, E_{i_1}, w_2, E_{i_2}, \dots, w_{s-1}, E_{i_{s-1}}, w_s, E_{i_s})$$

such that $w_n = E_{i_{n-1}} \cap E_{i_n}$ is a wedge for $n = 1, 2, \dots, s$. If we have another chain

$$c_1 = (E_{i_s}, w_{s+1}, E_{i_{s+1}}, w_{s+2}, E_{i_{s+2}}, \dots, w_{t-1}, E_{i_{t-1}}, w_t, E_{i_t})$$

starting at E_{i_s} , we can compose the two chains to obtain

$$c * c_1 = (E_{i_0}, w_1, E_{i_1}, w_2, E_{i_2}, \dots, w_{t-1}, E_{i_{t-1}}, w_t, E_{i_t}).$$

Let us consider a complex number $\mu \neq 0$. The *transformed number* $c(\mu)$ of by the chain c is defined as follows. If $s = 0$ we put $c(\mu) = \mu$. Put

$$c = c_{s-1} * (E_{i_{s-1}}, w_s, E_{i_s})$$

where c_{s-1} has length $s - 1$. For $\alpha = \text{Ind}(\mathcal{F}, E_{i_s}; w_s)$, we define

$$c(\mu) = -\alpha c_{s-1}(\mu).$$

Let us denote c^{-1} the chain obtained by reversing the order in c . By Remark 6 we have that

$$(7) \quad c^{-1}(c(\mu)) = \mu; \quad c(c^{-1}(\mu)) = \mu.$$

Lemma 1. *Consider a (circular) chain*

$$c = (E_{i_0}, w_1, E_{i_1}, w_2, E_{i_2}, \dots, w_{s-1}, E_{i_{s-1}}, w_s, E_{i_s})$$

such that $E_{i_0} = E_{i_s}$. For any $\mu \neq 0$ we have $c(\mu) = \mu$.

Proof. In view of Equation 7 the result is true if and only if it is true for one of the shifted chains

$$c_j = (E_{i_j}, w_j, E_{i_{j+1}}, w_{j+1}, E_{i_{j+2}}, \dots, w_s, E_{i_s} = E_{i_0}, w_1, E_{i_1}, \dots, E_{i_{j-1}}, w_{j-1}, E_{i_j}).$$

We do induction on the number of vertices of the graph \mathcal{G}_N and the length of c . If we have only one vertex, we are done. Let v be the last vertex incorporated to the construction of \mathcal{G} . If this vertex v does not appear in c , we are done by induction. Assume that v appears in c .

If v is a connected component of \mathcal{G} , we have only one vertex in c and we are done. If v is not isolated, we have three possibilities:

- (1) v is connected with exactly one vertex v_1 with a wedge w'_1 .
- (2) v is connected with two vertices v_1, v_2 by means of respective wedges w'_1, w'_2 . In this case v_1 and v_2 are connected by wedge \tilde{w}_{12} .
- (3) v is connected with three vertices v_1, v_2, v_3 by means of respective wedges w'_1, w'_2, w'_3 . In this case v_1, v_2, v_3 are connected two by two by wedges $\tilde{w}_{12}, \tilde{w}_{13}, \tilde{w}_{23}$.

In case (1), up to performing a shift of c , we may assume that c has the form

$$c = (v, w'_1, v_1, w_2, \dots, w_{s-1}, v_1, w'_1, v)$$

and we are done by induction applied to $c' = (v_1, w_2, \dots, w_{s-1}, v_1)$ as follows. Let $\alpha = \text{Ind}(\mathcal{F}, v; w'_1)$, put $\mu' = -\alpha\mu$, then we have

$$c(\mu) = (-1/\alpha)c'(\mu') = (-1/\alpha)\mu' = \mu.$$

In case (2), up to interchanging the role of v_1 and v_2 the appearance of v may be in one of the following two forms,

$$\begin{aligned} c &= c_1 * (v_1, w'_1, v, w'_1, v_1) * c_2, \\ c &= c_1 * (v_1, w'_1, v, w'_2, v_2) * c_2. \end{aligned}$$

The first one is treated as in the previous case. Assume we have the second one. Let us denote

$$\alpha = \text{Ind}(\mathcal{F}, v_1; \tilde{w}_{12}); \quad \beta = \text{Ind}(\mathcal{F}, v_1; \tilde{w}'_1); \quad \rho = \text{Ind}(\mathcal{F}, v_2; \tilde{w}'_1).$$

We know that $\beta = -\alpha\rho$. Consider the circular chain

$$\tilde{c} = c_1 * (v_1, \tilde{w}_{12}, v_2) * c_2.$$

In view of the fact that

$$(v_1, \tilde{w}_{12}, v_2)(\tilde{\mu}) = -\alpha\tilde{\mu} = (-\beta)(-1/\rho)\tilde{\mu} = (v_1, w'_1, v, w'_2, v_2)(\tilde{\mu}),$$

we deduce that $\tilde{c}(\mu) = c(\mu)$ and we are done since by induction we have $\tilde{c}(\mu) = \mu$.

Case (3) is treated as the previous one. \square

Now we go to the proof of Proposition 7. Since the compact part of the partial separatrix C is connected, we can join a generic point p_1 in Γ_1 with a generic point p_2 in Γ_2 by a real path γ . Moreover γ may be chosen in such a way that it produces only finitely many changes of irreducible curves in C . The connected change of (trace) irreducible curves of C gives a transition of invariant compact component of the divisor. In this way, we obtain a circular chain

$$c = (E_{i_0}, w_1, E_{i_1}, w_2, E_{i_2}, \dots, w_{s-1}, E_{i_{s-1}}, w_s, E_{i_s} = E_{i_0})$$

such that if μ is the index for Γ_1 then $c(\mu)$ is the index for Γ_2 . By Lemma 1 we have that $c(\mu) = \mu$ and the proof is ended.

Remark 7. Corollary 2 may now be reformulated by stating that

$$\left(\sum_{\gamma \in \mathcal{B}_i C} (\gamma, \Gamma)_p \right) \text{Ind}(C; E_i) = -\alpha \left(\sum_{\eta \in \mathcal{B}_j C} (\eta, \Gamma)_p \right) \text{Ind}(C; E_j).$$

7. REAL SADDLES AT INCOMPLETE POINTS

Here we give a result relating incomplete points and real saddle curves. This is a key point in the proof of Theorem 1.

Proposition 8. *Let p be an incomplete point belonging to a compact invariant component E_i^k . There is a compact curve $\Gamma \subset \text{Sing} \mathcal{F}_k$ with $\Gamma \subset E_i^k$ such that Γ is not a real saddle.*

Proof. As usual we do induction on $N - k$. If $k = N$ there is nothing to prove, since p is a complete point. Assume that $k < N$. We assume without loss of generality that $p \in Y_k$. Moreover, since p is an incomplete point, we necessarily have that $Y_k = \{p\}$ in view of Remark 3. Now, it is enough to find $\gamma \in \mathcal{B}_i^k(p)$ such that

$$\text{Ind}(\mathcal{F}, E_i; \gamma) \notin \mathbb{R}_{<0}.$$

We assume by contradiction that all $\gamma \in \mathcal{B}_i^k(p)$ are real saddle curves.

First case: π_{k+1} is a dicritical blow-up. We apply Proposition 6 to the dicritical component E_{k+1}^{k+1} to see that

$$(8) \quad \sum_{q \in L} \sum_{\gamma \in \mathcal{B}_i^k(p)} (\gamma', L)_q \text{Ind}(\mathcal{F}, E_i; \gamma) = \sum_{q \in L} \text{Ind}(\mathcal{F}|_{E_{k+1}^{k+1}}, L; q)$$

where $L = E_{k+1}^{k+1} \cap E_i^{k+1}$. The left hand side of Equation 8 is a negative number but the right hand side coincides with the self-intersection of L in E_{k+1}^{k+1} , that is, it has the value $+1$. This is the desired contradiction.

Second case: π_{k+1} is a non dicritical blow-up. Put $L = E_{k+1}^{k+1} \cap E_i^{k+1}$ as before and $\alpha = \text{Ind}(\mathcal{F}, E_i; L)$. Let us consider a generic plane Δ at p and $\mathcal{G} = \mathcal{F}_k|_{\Delta}$. The blow-up π_{k+1} induces a blow-up $\tilde{\Delta} \rightarrow \Delta$ and the transform of \mathcal{G} by this blow-up is $\tilde{\mathcal{G}} = \mathcal{F}_{k+1}|_{\tilde{\Delta}}$. By the properties of the indices of Camacho-Sad we have

$$\sum_{q \in \tilde{\Delta} \cap E_{k+1}^{k+1}} \text{Ind}(\tilde{\mathcal{G}}, \tilde{\Delta} \cap E_{k+1}^{k+1}; q) = -1.$$

Moreover, we have

$$\sum_{q \in \tilde{\Delta} \cap E_{k+1}^{k+1}} \text{Ind}(\tilde{\mathcal{G}}, \tilde{\Delta} \cap E_{k+1}^{k+1}; q) = \text{Ind}(\mathcal{F}, E_{k+1}; L) + \sum_{\Lambda \subset E_{k+1}^{k+1}, \Lambda \neq L} d_{\Lambda} \text{Ind}(\mathcal{F}, E_{k+1}; \Lambda).$$

We know that $\text{Ind}(\mathcal{F}, E_{k+1}; L) = 1/\alpha$ and by Proposition 6 we have

$$\sum_{\Lambda \subset E_{k+1}^{k+1}, \Lambda \neq L} d_\Lambda \text{Ind}(\mathcal{F}, E_{k+1}; \Lambda) = -\frac{1}{\alpha} \left\{ \sum_{q \in E_{k+1}^{k+1}} \sum_{\gamma \in \mathcal{B}_i^k(p)} (\gamma, L)_q \text{Ind}(\mathcal{F}, E_i; \gamma) \right\}.$$

That is

$$-1 = \frac{1}{\alpha} - \frac{r}{\alpha}$$

with $r < 0$. Thus $\alpha = (r - 1)$ is a negative real number and L is a real saddle. By induction hypothesis, all the points in L must be complete, since otherwise the non real saddle in E_i^{k+1} is not L and projects to a non real saddle in E_i^k . Moreover, since the blow-up is non dicritical, there is at least one incomplete point $q \in E_{k+1}^{k+1}$. Let Θ be a curve through q that is not a real saddle and consider a point $p' \in \Theta \cap L$. If Θ is a trace curve, by the transition of indices at complete points given in Corollary 2 we deduce the existence of a trace curve $\tilde{\Theta} \subset E_i^{k+1}$ such that $\tilde{\Theta} \subset E_i^{k+1}$ is not a real saddle. If Θ is contained in the intersection of two divisors, by Corollary 1 we find a non real saddle $\tilde{\Theta} \subset E_i^{k+1}$ with $\tilde{\Theta} \not\subset E_{k+1}^{k+1}$. The projection of $\tilde{\Theta}$ gives the desired contradiction. \square

8. UNINTERRUPTED NODAL COMPONENTS

Let us recall the notion of uninterrupted nodal component introduced in [11]. By definition, an *uninterrupted nodal component* of \mathcal{F}_N, E^N is a connected union \mathcal{N} of irreducible curves $\Gamma \subset \text{Sing} \mathcal{F}_N$ satisfying the following conditions:

- (1) Each $\Gamma \subset \mathcal{N}$ is a *nodal curve* (see Definition 4).
- (2) The component \mathcal{N} is *uninterrupted* in the sense that there are exactly two curves Γ_1 and Γ_2 in \mathcal{N} through any point $p \in \mathcal{N}$ of dimensional type three.

Recall that an uninterrupted nodal component \mathcal{N} is *incomplete* if and only if it intersects at least one compact dicritical component of the exceptional divisor E^N . Otherwise, we say that \mathcal{N} is *complete*.

The next result shows the compatibility between the uninterrupted nodal components and the partial separatrices, in the last step of the reduction of singularities.

Proposition 9 (Global trace transitions). *Let \mathcal{N} be a uninterrupted nodal component. Consider a partial separatrix C and a compact invariant component E_i of the exceptional divisor E . If there is $\Gamma_0 \in \mathcal{B}_i C$ with $\Gamma_0 \subset \mathcal{N}$ then $\Gamma \subset \mathcal{N}$ for any $\Gamma \in \mathcal{B}_i C$.*

Proof. The proof is similar to the proof of Proposition 7. Let us consider the dual graph \mathcal{G}_N as in Proposition 7. Take two curves $\Gamma_0, \Gamma_1 \in \mathcal{B}_i C$. We can connect Γ_0, Γ_1 by a circular chain

$$c = (E_i = E_{i_0}, w_1, E_{i_1}, w_2, E_{i_2}, \dots, w_{s-1}, E_{i_{s-1}}, w_s, E_{i_s} = E_i)$$

as in Lemma 1. Now, let us recall that at a point of dimensional type three we have either no curves of \mathcal{N} or exactly two of them. In this way, we have the following rule of behavior for the curves $\Gamma_{i_j} \subset E_{i_j} \cap C$ that we are considering in the chain c :

- (1) If $\Gamma_{i_{j-1}} \subset \mathcal{N}$ and $w_j \not\subset \mathcal{N}$, then $\Gamma_{i_j} \subset \mathcal{N}$.
- (2) If $\Gamma_{i_{j-1}} \subset \mathcal{N}$ and $w_j \subset \mathcal{N}$, then $\Gamma_{i_j} \not\subset \mathcal{N}$.
- (3) If $\Gamma_{i_{j-1}} \not\subset \mathcal{N}$ and $w_j \subset \mathcal{N}$, then $\Gamma_{i_j} \subset \mathcal{N}$.
- (4) If $\Gamma_{i_{j-1}} \not\subset \mathcal{N}$ and $w_j \not\subset \mathcal{N}$, then $\Gamma_{i_j} \not\subset \mathcal{N}$.

Let us denote $\epsilon(w_{i_j}) = -1$ if $w_{i_j} \subset \mathcal{N}$ and $\epsilon(w_{i_j}) = 1$ otherwise. Now, it is enough to prove that

$$\epsilon(w_{i_1})\epsilon(w_{i_2}) \cdots \epsilon(w_{i_s}) = 1.$$

This can be done by the same arguments as in the proof of Lemma 1. \square

Now, we consider an intermediate step (M_k, F_k) of the reduction of singularities and we will study the transition properties of a *fixed complete uninterrupted nodal component* \mathcal{N} at this level k (see also [11]). We put

$$\mathcal{N}_k = \rho_k(\mathcal{N}).$$

Note that $\mathcal{N}_k \cap F_k$ is either a single point or a finite union of compact curves.

Proposition 10 (Triple points transitions). *Let $p \in F_k$ be a point belonging to three compact components E_i^k, E_j^k and E_ℓ^k of E^k . Assume that $\Gamma_\ell = E_i^k \cap E_j^k \subset \mathcal{N}_k$. Then E_i^k, E_j^k and E_ℓ^k are invariant and*

$$\Gamma_j \subset \mathcal{N}_k \Leftrightarrow \Gamma_i \not\subset \mathcal{N}_k,$$

where $\Gamma_j = E_\ell^k \cap E_i^k$ and $\Gamma_i = E_\ell^k \cap E_j^k$.

Proof. We do induction on $N - k$. If $k = N$ we are done by the definition of complete uninterrupted nodal component.

Assume that $k < N$ and $p \in Y_k$ as usual. Since p is in the intersection of three compact components then $Y_k = \{p\}$. Denote

$$p'_u = E_{k+1}^{k+1} \cap \Gamma'_u, \quad L'_u = E_{k+1}^{k+1} \cap E_u^{k+1}; \quad u \in \{i, j, \ell\}.$$

By induction on p'_ℓ we have that E_i^{k+1}, E_j^{k+1} and E_{k+1}^{k+1} are invariant. In particular π_{k+1} is non dicritical. We also have that either L'_i or L'_j are contained in \mathcal{N}_{k+1} . Now by induction on p'_j or p'_i respectively, we deduce that E_ℓ^{k+1} and hence E_ℓ^k is invariant.

Assume now that $L'_i \subset \mathcal{N}_{k+1}$ and hence $L'_j \not\subset \mathcal{N}_{k+1}$. By induction on p'_j we have two possibilities:

- (1) $\Gamma'_j \subset \mathcal{N}_{k+1}$ and $L'_\ell \not\subset \mathcal{N}_{k+1}$. It is not possible to have that $\Gamma'_i \subset \mathcal{N}_{k+1}$ since at p'_i we have the two other corner curves not in \mathcal{N}_{k+1} .
- (2) $\Gamma'_j \not\subset \mathcal{N}_{k+1}$ and $L'_\ell \subset \mathcal{N}_{k+1}$. Then we have that $\Gamma'_i \subset \mathcal{N}_{k+1}$ since $L'_j \not\subset \mathcal{N}_{k+1}$.

We conclude in the same way in the case that $L'_j \subset \mathcal{N}_{k+1}$ and $L'_i \not\subset \mathcal{N}_{k+1}$. \square

Proposition 11 (Trace transitions). *Let C be a partial separatrix. Consider a point $p \in C_k$ complete for C and belonging to a compact invariant component E_i^k . Suppose that there is $\gamma \in \mathcal{B}_i^k(C; p)$ with $\gamma \subset \mathcal{N}_k$ and that p belongs to another compact component E_j^k . We have:*

- (1) E_j^k is an invariant component.
- (2) Put $\Gamma = E_i^k \cap E_j^k$. Then one of the following statements holds:
 - (a) $\Gamma \subset \mathcal{N}_k$ and any $\tilde{\gamma} \in \mathcal{B}_j^k(C; p)$ is a real saddle.
 - (b) Γ is a real saddle and for any $\tilde{\gamma} \in \mathcal{B}_j^k(C; p)$ we have $\tilde{\gamma} \subset \mathcal{N}_k$.

Proof. As usual, we do induction on $N - k$. If $k = N$, we are done. Indeed, p does not belong to any dicritical compact component, since $p \in \mathcal{N}$ and \mathcal{N} is complete. Moreover, the alternative in (2) means that \mathcal{N} is uninterrupted.

Assume now that $k < N$ and $p \in Y_k$ as usual. If Y_k is a curve, there is only one compact component of E^k through p and E_j^k does not exist. We assume thus that $Y_k = \{p\}$. Since $p \in C_k$ is a complete point for C , then π_{k+1} is non dicritical. Let us denote

$$L'_i = E_{k+1}^{k+1} \cap E_i^{k+1}, \quad L'_j = E_{k+1}^{k+1} \cap E_j^{k+1}; \quad \Gamma' = E_i^{k+1} \cap E_j^{k+1}$$

and let p' be the intersection point $E_{k+1}^{k+1} \cap \Gamma'$.

Let us see that E_j^{k+1} is invariant. If $L'_i \subset \mathcal{N}$, then E_j^{k+1} is invariant by Proposition 10 applied at p' . If $L'_i \not\subset \mathcal{N}$, the strict transform γ' of γ intersects L'_i at some points. Let q be one of such points. By induction hypothesis on q there is a curve $\gamma^* \subset E_{k+1}^{k+1}$ with $\gamma^* \subset \mathcal{N}_{k+1}$ corresponding to the same partial separatrix C . We conclude that E_j^{k+1} is invariant by induction hypothesis applied at the points of $\gamma^* \cap E_j^{k+1}$.

Now, assume that $\Gamma \subset \mathcal{N}_k$. Hence $\Gamma' \subset \mathcal{N}_{k+1}$. By Proposition 10, we have two possible situations:

- i) $L'_i \subset \mathcal{N}_{k+1}$ and $L'_j \not\subset \mathcal{N}_{k+1}$.
- ii) $L'_j \subset \mathcal{N}_{k+1}$ and $L'_i \not\subset \mathcal{N}_{k+1}$.

Assume we have i). Consider a point $q \in \gamma' \cap L'_i$. By induction hypothesis at q , there is a curve $\gamma^* \in \mathcal{B}_{k+1}^{k+1}(C; q)$ such that $\gamma^* \not\subset \mathcal{N}_{k+1}$. We consider a point $q' \in \gamma^* \cap L'_j$. Also by induction hypothesis

at q' there is $\tilde{\gamma}' \in \mathcal{B}_j^{k+1}(C; q')$ such that $\tilde{\gamma}' \not\subset \mathcal{N}_{k+1}$. Now it is enough to take $\tilde{\gamma}$ the image of $\tilde{\gamma}'$ by π_{k+1} . In the case ii) we do a similar argumentation.

Also, if $\Gamma \not\subset \mathcal{N}_k$ we have two possibilities:

- i) $L'_i \subset \mathcal{N}_{k+1}$ and $L'_j \subset \mathcal{N}_{k+1}$.
- ii) $L'_j \not\subset \mathcal{N}_{k+1}$ and $L'_i \not\subset \mathcal{N}_{k+1}$.

By the same kind of argumentations we find $\tilde{\gamma} \in \mathcal{B}_j^k(C; p)$ with $\tilde{\gamma} \subset \mathcal{N}_k$.

The statements relative to the real saddles are consequence of Proposition 6. \square

9. INCOMPLETENESS OF UNINTERRUPTED NODAL COMPONENTS

As explained in the Introduction, Theorem 1 is a consequence of the following result:

Theorem 2. *Any uninterrupted nodal component \mathcal{N} of \mathcal{F}_N, E^N is incomplete.*

In this section we provide a proof for Theorem 2. We assume that \mathcal{F} has no germ of invariant surface and that \mathcal{N} is a *complete* uninterrupted nodal component. We shall find a contradiction with the fact that \mathcal{N} is complete.

Let $b > 0$ be the *date of birth of the compact part of \mathcal{N}* , that is we assume that $\mathcal{N}_k \cap F_k$ is a single point for $0 \leq k < b$ and that \mathcal{N}_b contains at least one compact curve. Note that \mathcal{N} contains at least one compact curve, because π_1 is the blow-up centered at the origin and hence the fiber $F = \pi^{-1}(0)$ is the union of the compact components of E . If we take a point $q \in \mathcal{N} \cap F$, the compact components of E through q are invariant, by the completeness of \mathcal{N} . If the dimensional type of q is two, the singular locus of \mathcal{F}_N coincides locally with \mathcal{N} and it is contained in the invariant components of E through p . If the dimensional type is three, we have two curves of \mathcal{N} , one of them is necessarily contained in a compact invariant component. As a consequence of this we find that $1 \leq b \leq N$.

Lemma 2. *The blow-up π_b is not centered at a germ of curve.*

Proof. Suppose by contradiction that π_b is the (monoidal) blow-up centered at a germ of curve (Y_{b-1}, p) . The point $p \in F_{b-1}$ is contained in a compact component E_i^{b-1} of E^b transversal to Y_{b-1} . Now, we have equireduction along Y_{b-1} . We consider all the blow-ups we do over Y_{b-1} and we reach a desingularized situation over the point p . The fiber of p contains a maximal connected union of compact curves in \mathcal{N} , say

$$\Gamma_{j_1} \cup \Gamma_{j_2} \cup \dots \cup \Gamma_{j_s}, \quad \Gamma_{j_\ell} \cap \Gamma_{j_{\ell+1}} \neq \emptyset, \quad \ell = 1, 2, \dots, s-1.$$

Each Γ_{j_ℓ} is of the form $\Gamma_{j_\ell} = E_i \cap E_{j_\ell}$ where E_{j_ℓ} is non compact. Moreover, by the fact that \mathcal{N} is uninterrupted, we have two possibilities:

- (1) The curves Γ_{j_ℓ} represent all the components of E contained in the inverse image of Y_{b-1} .
- (2) There are two noncompact curves $\gamma_1 = E_{j_1} \cap E_{j_0}$ and $\gamma_s = E_{j_s} \cap E_{j_{s+1}}$ such that $\gamma_1, \gamma_s \subset \mathcal{N}$ and none of the curves $E_{j_\ell} \cap E_{j_{\ell+1}}$ are in \mathcal{N} for $\ell = 1, 2, \dots, s-1$.

Moreover the concerned divisors are non dicritical. Now, we can apply the refined Camacho-Sad Theorem [20] to a transversal plane section at a generic point of Y_{b-1} near p . In this way, we find a non compact trace curve of generic index not in $\mathbb{R}_{>0}$ that cuts one of the compact curves Γ_{j_ℓ} . Since \mathcal{N} is uninterrupted there is a compact trace curve of \mathcal{N} contained in E_i^{b-1} , this is the desired contradiction. \square

In view of Lemma 2 we suppose that π_b is a (quadratic) blow-up centered at the point p . We also know that π_b is non dicritical, since there is a compact curve $\Gamma \subset E_b^b \cap \mathcal{N}_b$. Moreover, the point p belongs only to compact components of E^{b-1} , otherwise the blow-up should be monoidal.

We consider separately the cases $b = 1$ and $b > 1$.

Assume that $b = 1$ and consider the exceptional divisor $E_1^1 = \pi_1^{-1}(0)$. The curve $\Gamma \subset \mathcal{N}_1 \cap E_1^1$ is a compact trace curve and thus there is a partial separatrix $C = C_\Gamma$ with $\Gamma \subset C_1$. Consider the set $C_1 \cap E_1^1$. In view of Proposition 9, any irreducible component Γ' of $C_1 \cap E_1^1$ satisfies $\Gamma' \subset \mathcal{N}_1$. Recall that C is an incomplete partial separatrix in view of Proposition 2 and thus there is at least a

point $q \in C_1 \cap E_1^1$ such that q is incomplete for C by Proposition 5. Therefore, we have the following situation for $k = 1$:

A(k): There is a compact trace curve $\Gamma \subset \mathcal{N}_k$ and a point $q \in \Gamma$ incomplete for the partial separatrix C_Γ .

Assume now that $b > 1$. In this case p belongs to at least one compact component of E^{b-1} . Recall that all the components of E^{b-1} through p are compact. We discuss case by case.

(1). We have only one component E_i^{b-1} of E^{b-1} through p , which may be dicritical or invariant.

(1-a). Assume that E_i^{b-1} is dicritical. We perform the blow-up π_b and obtain a trace curve $\Gamma \subset \mathcal{N}_b \cap E_b^b$. By Proposition 11, the points in $\Gamma \cap E_i^b$ are incomplete for C_Γ . We arrive to situation **A(k)** for $k = b$.

(1-b). Let us suppose now that E_i^{b-1} is invariant. Consider the projective line $L = E_i^b \cap E_b^b$, then either $L \subset \mathcal{N}_b$ or not.

(1-b-1). Assume that $L \subset \mathcal{N}_b$. By taking a generic plane section at p and by Camacho-Sad's argument on the sum of indices as in the proof of Proposition 8, we find a compact trace curve $\Theta \subset E_b^b$ such that the index of Θ is not in $\mathbb{R}_{>0}$. We consider a point q of intersection of L and Θ . If q is a complete point for C_Θ , by Proposition 11 we should obtain a trace curve in E_i^b contained in \mathcal{N}_b ; this is not possible since b is the date of birth of \mathcal{N} . Thus q is an incomplete point. We obtain the following situation for $k = b$:

B(k): There are a compact curve $\Gamma \subset \mathcal{N}_k$ such that $\Gamma = E_i^k \cap E_j^k$ is the intersection of two invariant compact components and an incomplete point $q \in \Gamma$.

(1-b-2). Assume that $L \not\subset \mathcal{N}_b$. Then there is a trace curve $\Gamma \subset \mathcal{N}_b \cap E_b^b$. Consider a point $q \in \Gamma \cap L$. If q is complete for C_Γ , we apply the trace transitions of Proposition 11 and this contradicts the fact that b is the date of birth of \mathcal{N} . Thus the point q is incomplete for C_Γ and we arrive to situation **A(k)** for $k = b$.

(2). There are two components E_i^{b-1}, E_j^{b-1} of E^{b-1} through p .

(2-a). If both components are dicritical, we do an argument as in case (1-a) to obtain **A(k)** for $k = b$.

(2-b). If E_i^{b-1} is invariant and E_j^{b-1} is dicritical, we have two possibilities:

(2-b-1). There is a trace curve $\Gamma \subset \mathcal{N}_b \cap E_b^b$. Take a point $q \in \Gamma \cap E_j^b$, in view of Proposition 11, the point q must be incomplete for C_Γ . We obtain **A(k)** for $k = b$.

(2-b-2). The other case is that $L = E_i^b \cap E_b^b$ is contained in \mathcal{N} . By Proposition 10 this is not possible.

(2-c). Both E_i^{b-1} and E_j^{b-1} are invariant.

(2-c-1). If $E_i^b \cap E_b^b \subset \mathcal{N}_b$, by Proposition 10 we have that $E_j^b \cap E_b^b \subset \mathcal{N}_b$. By the already used argument on the sum of indices for a generic plane section at p , we find a trace curve $\Theta \subset E_b^b$ of index not in $\mathbb{R}_{>0}$. The trace transitions of Θ described in Proposition 11 will produce curves of \mathcal{N}_{b-1} previously existing in $E_i^{b-1} \cup E_j^{b-1}$, unless we have incomplete points in $E_i^b \cap E_b^b$ and $E_j^b \cap E_b^b$ at the intersections with Θ . Thus we obtain **B(k)** for $k = b$.

(2-c-2). Assume now that $E_i^b \cap E_b^b \not\subset \mathcal{N}_b$. By Proposition 10 we also have that $E_j^b \cap E_b^b \not\subset \mathcal{N}_b$, since otherwise we should have $E_i^{b-1} \cap E_j^{b-1} \subset \mathcal{N}_{b-1}$. The other possible curves in E_b^b are of trace type and thus the curve $\Gamma \subset \mathcal{N}_b$ that appears after π_b is a trace curve. We obtain **A(k)** for $k = b$ as in (1-b-2).

(3). There are three components E_i^{b-1}, E_j^{b-1} and E_ℓ^{b-1} of E^{b-1} through p . We use the same kind of argumentation as in the cases (1) and (2) to reach one of the situations **A(k)** or **B(k)** for $k = b$.

We have identified two situations **A(k)** and **B(k)** such that one of them appears in the birth level of the uninterrupted nodal component \mathcal{N} . We would like to show the persistency of this phenomenon at further levels of the reduction of singularities. However, another situation must be considered, which is the following:

C(k): There is a compact invariant component E_i^k , a compact curve $\Gamma \subset E_i^k \cap \mathcal{N}_k$ and an incomplete point $q \in \Gamma$ such that the following holds: every global irreducible curve $\Theta \subset \text{Sing}\mathcal{F}_k \cap E_i^k$ with $q \in \Theta$ is either in \mathcal{N}_k or a real saddle.

Proposition 12 (Persistence). *Assume that there is an index $1 \leq k < N$, a global curve $\Gamma \subset \mathcal{N}_k$ and an incomplete point $q \in \Gamma$ in one of the situations **A(k)**, **B(k)** or **C(k)**. Then there is a global curve $\Gamma' \subset \mathcal{N}_{k+1}$ and an incomplete point $q' \in \Gamma'$ in one of the situations **A(k+1)**, **B(k+1)** or **C(k+1)**.*

Proof. If π_{k+1} is centered at Y_k with $q \notin Y_k$, we obviously reach **A(k+1)**, **B(k+1)** or **C(k+1)** at the “same” point q . Thus, we assume $q \in Y_k$. Moreover, since q is incomplete, we have $Y_k = \{q\}$. Indeed, if Y_k is a germ of curve, the point q is complete. Then $E_{k+1}^{k+1} = \pi_k^{-1}(q)$ is a projective plane.

(a). Assume that we have **A(k)**. Let E_i^k be the compact invariant component such that $\Gamma \subset E_i^k$. We consider two cases:

(a-1). *The blow-up π_{k+1} is dicritical.* We consider the strict transform Γ' of Γ and a point $q' \in \Gamma' \cap E_{k+1}^{k+1}$. In view of Proposition 11 the point q' must be incomplete for C_Γ and we recover the situation **A(k+1)**.

(a-2). *The blow-up π_{k+1} is non dicritical.* Let us put $L = E_{k+1}^{k+1} \cap E_i^{k+1}$.

(a-2-1). Assume first that $L \subset \mathcal{N}_{k+1}$. If there is an incomplete point $q' \in L$ we obtain **B(k+1)**. Thus we assume that all the points in L are complete. We find an incomplete point $q' \in E_{k+1}^{k+1}$. By Proposition 8, there is a global irreducible curve $\Gamma' \subset E_{k+1}^{k+1}$ with $q' \in \Gamma'$ that is not a real saddle. We consider a (complete) point $p' \in L \cap \Gamma'$. Now, by Proposition 11 or Proposition 10 we see that Γ' must be contained in \mathcal{N}_{k+1} . This argument also works for all non real saddle curves through q' . Hence we find **C(k+1)** or **B(k+1)** at q' .

(a-2-2). It remains to consider the case that $L \not\subset \mathcal{N}_{k+1}$. If there is a point $q' \in L \cap \tilde{\Gamma}$ incomplete for C_Γ , where $\tilde{\Gamma}$ is the strict transform of Γ , we obtain **A(k+1)** at q' . If not, we consider the transitions given in Proposition 11 to see that $C_\Gamma \cap E_{k+1}^{k+1}$ is contained in \mathcal{N}_{k+1} . Moreover, there exists a point $q' \in C_\Gamma \cap E_{k+1}^{k+1}$ incomplete for C_Γ . We recover **A(k+1)** at q' .

(b). Assume we have **B(k)**. Put $L_i = E_{k+1}^{k+1} \cap E_i^{k+1}$ and $L_j = E_{k+1}^{k+1} \cap E_j^{k+1}$. Let $p' = L_i \cap L_j$. By Proposition 10 we know that π_{k+1} is a non dicritical blow-up. Moreover, we have that

$$L_i \subset \mathcal{N}_{k+1} \Leftrightarrow L_j \not\subset \mathcal{N}_{k+1}.$$

To fix ideas, suppose that $L_i \subset \mathcal{N}_{k+1}$ and $L_j \not\subset \mathcal{N}_{k+1}$. If there is an incomplete point at L_i we have **B(k+1)** at such a point. So we assume that all the points in L_i are complete. This means that there is an incomplete point $q' \in E_{k+1}^{k+1} \setminus L_i$. We repeat at this point the previous argument for the case (a-2-1) and we recover **C(k+1)** at q' .

(c). Let us assume finally that we have **C(k)**. We also suppose that we are not in the situations **A(k)** or **B(k)** already studied and hence Γ is a trace curve and q is complete for C_Γ . As in case (b), we may assume that π_{k+1} is non dicritical, otherwise we obtain **A(k+1)** at the strict transform of Γ . Let us put $L = E_{k+1}^{k+1} \cap E_i^{k+1}$.

(c-1). Assume first that $L \subset \mathcal{N}_{k+1}$. We may assume that all the points in L are complete, otherwise we get **B(k+1)**. All the global irreducible curves $\Theta \subset E_{k+1}^{k+1}$ with $\Theta \neq L$ are either real saddles or curves in \mathcal{N}_{k+1} in view of Propositions 11 and 10. On the other hand, we necessarily have an incomplete point $q' \in E_{k+1}^{k+1}$. The non real saddle passing through q' given by Proposition 8 is then contained in \mathcal{N}_{k+1} , as well as any other non real saddle curve. Thus, we recover **C(k+1)** at q' .

(c-2). Let us assume that $L \not\subset \mathcal{N}_{k+1}$. Let $\Gamma' \subset \mathcal{N}_{k+1}$ be the strict transform of Γ and take a complete point $p \in \Gamma' \cap L$. By the transition rules in Proposition 11 we obtain that L is a real saddle. If there is an incomplete point $q' \in L$ we are done, since it satisfies **A(k+1)**. We suppose that all the points in L are complete and we take an incomplete point $q' \in E_{k+1}^{k+1} \setminus L$. Let us see that all the global irreducible curves $\Theta \subset E_{k+1}^{k+1} \cap \text{Sing}\mathcal{F}_{k+1}$ are either real saddles or contained in \mathcal{N}_{k+1} . In this way we obtain **C(k+1)** at q' and we are done. We look at the transitions through L at $\Theta \cap L$ described in Corollary 2. Recalling that the curves in $E_i^{k+1} \cap \text{Sing}\mathcal{F}_{k+1}$ arriving at L are either real saddle curves or in \mathcal{N}_{k+1} , we see that Θ is also in \mathcal{N}_{k+1} or a real saddle curve. \square

As a consequence of Proposition 12 we arrive to **A**, **B** or **C** in the final step, which is not possible since all the points in the final step are complete points. This is the desired contradiction. Thus, the only possibility is that there are no complete uninterrupted nodal components. This ends the proof of Theorem 2.

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